On the Angles of the Regular Polytopes of Four-dimensional Space.

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§ 1. *Notations.*

We introduce general notations for the angles in question, indicating

by a a plane angle between edges (of polygons),
" b " " " faces (of polyhedra),
" c " " " bounding bodies (of polytopes),

these angles varying from naught to $360° = 2\pi$;

by α a solid angle between faces (of polyhedra),
" β " " round an edge (of polytopes),

these angles varying from naught to $720° = 4\pi$;

by A a four-dimensional angle (of polytopes),

this angle varying from naught to $1440° = 8\pi$.

§ 2. *Relations between a, b, c, α, β.*

By means of an isosceles spherical triangle, or one of its two rectangular halves, we can express b in a and c in b by using the cosine-formula, and α in b and β in c by using the expression for the spherical excess. But to that end we ought to know the number p_3 of faces concurring in a vertex for the bounding polyhedron and the number p_4 of limiting polyhedra passing through an edge of the polytope itself. If we add the number p_2 of vertices (and sides) of the limiting polygons of the polytope, we have $a = \pi - \dfrac{2\pi}{p_2}$; so by the relations we are going to deduce all the angles a, b, c, α, β can be expressed in p_2, p_3, p_4.

Relations (a, b) *and* (α, b).—The three edges of the trihedral angle corresponding to the spherical triangle in view are two adjacent edges of the regular polyhedron and the line joining their common point to the centre of gravity of the polyhedron. So the three angles of the spherical triangle are $\frac{b}{2}, \frac{b}{2}, \frac{2\pi}{p_3}$, whilst the side opposite to $\frac{2\pi}{p_3}$ is a. So the cosine-formula for the angles gives

$$\cos\frac{2\pi}{p_3} = -\cos^2\frac{b}{2} + \sin^2\frac{b}{2}\cos a,$$

or

$$\sin\tfrac{1}{2}b = \frac{\cos\frac{\pi}{p_3}}{\cos\frac{a}{2}}, \tag{1}$$

and from the spherical excess $b + \frac{2\pi}{p_3} - \pi$ taken p_3 times we deduce

$$\alpha = p_3 b - (p_3 - 2)\pi. \tag{2}$$

Relations (b, c) *and* (β, c).—Here the three edges of the trihedral angle are the lines joining the midpoint of an edge to the centres of gravity of two adjacent faces through that edge and to the centre of gravity of the polytope itself. So the three angles are $\frac{c}{2}, \frac{c}{2}, \frac{2\pi}{p_4}$; the side opposite to $\frac{2\pi}{p_4}$ is b; so we find

$$\sin\tfrac{1}{2}c = \frac{\cos\frac{\pi}{p_4}}{\cos\frac{b}{2}}, \tag{3}$$

$$\beta = p_4 c - (p_4 - 2)\pi. \tag{4}$$

So, if the number of sides and vertices of the polygons of the polytope is represented by p_2, the formulae (1), (2), (3), (4) enable us indeed to express all the angles a, b, c, α, β in p_2, p_3, p_4.

§ 3. *The Angle A.*

For the first time the angle A was calculated for all the regular polytopes by L. Schläfli in 1852.* We will indicate here in what manner he derived

* See the posthumous work "Theorie der vielfachen Kontinuität" published by Mr. J. H. Graf, p. 118.

his results, though we must omit parts of the demonstration. His chief instrument is a differential formula. Let us consider the simplest case of an angle A formed by four lines l_1, l_2, l_3, l_4 meeting in O and not lying together in the same three-dimensional space. Let us represent by a_{12} the angle between l_1 and l_2, by c_{12} the "adjacent" angle between the spaces $(l_1, l_2)\, l_3$ and $(l_1, l_2)\, l_4$ meeting in the plane l_1, l_2. Then the differential formula under discussion is

$$\delta A = \frac{2}{\pi} \sum_1^6 a \delta c, \tag{I}$$

the unit 180° of A being an eighth part of the whole four-dimensional space round O and the summation Σ extending over the six angles $a_{1,2}, \ldots, a_{3,4}$ and the "adjacent" angles $c_{1,2}, \ldots, c_{3,4}$. Now let us assume any point O' within the angle A and drop perpendiculars from O' to the four limiting spaces of A; then another four-dimensional angle A' is formed, called the *supplementary* four-dimensional angle of A, the angles a' and c' of A' being the supplements of the corresponding angles c and a of A. For this new figure (I) becomes

$$\delta A' = \frac{2}{\pi} \sum_1^6 (\pi - c) \delta(\pi - a) = -\frac{2}{\pi} \sum_1^6 (\pi - c) \delta a. \tag{II}$$

Addition of (I) and (II) gives

$$\delta(A + A') = \frac{2}{\pi} \sum_1^6 \{a \delta c - (\pi - c) \delta a\} = -\frac{2}{\pi} \sum_1^6 \delta \{(\pi - c) a\},$$

which equation can be integrated. As A' becomes 720° when A disappears we get

$$A + A' = 720° - \frac{2}{\pi} \sum_1^6 (\pi - c) a. \tag{III}$$

This formula enables us to find the value of A in the case of the regular polytopes C_5, C_8, C_{120}, the four-dimensional angles of which admit only four edges. But it must be extended in the case of the other regular polytopes to what may be called "regular four-dimensional angles" with more than four edges. A four-dimensional angle is "regular" if there exists a line through its vertex with the property that the space normal to that line in any point P

cuts the edges in the vertices of a regular polyhedron having P for centre. Now it can be shown that the relation (III) holds for any regular four-dimensional angle and the supplementary regular one of the perpendiculars, if the summation Σ be extended to all the equal angles a and the corresponding equal angles c, the number of terms under the sign Σ being therefore equal to the number of edges of the regular polyhedron mentioned above. But if A is a four-dimensional angle of any regular polytope, and the vertex O' of the supplementary angle A' is the centre of that polytope, the edges of A' are the lines joining that centre O to the centres of gravity of the limiting bodies passing through the vertex O of A; so A' is the "central four-dimensional angle" of another regular polytope, viz., of the one polar-reciprocal to the original one. Now, if we designate—in accordance with my German text-book—by e, k, f, r the numbers of vertices, edges, faces, limiting bodies of the original regular polytope, these quantities taken in reversed order r, f, k, e represent at the same time the numbers of vertices, edges, faces, limiting bodies of the polar-reciprocal one. We find, therefore,

$$A' = \frac{1440^\circ}{e}.$$

Moreover, the number of terms under the sign Σ can be expressed in p_2, e, f. For $p_2 f$ represents how many times any vertex lies in any face, and therefore at the same time how many times any face passes through any vertex; so $\frac{p_2 f}{e}$ represents the number of faces passing through the vertex of A, *i. e.*, the number of edges of the regular polyhedron. So by introducing $\pi - \frac{2\pi}{p_2}$ for a we find, after slight reductions, the general result,*

$$A = \frac{1}{e}\left\{720^\circ(e-2) - 2f(p_2 - 2)(\pi - c)\right\}. \tag{5}$$

§ 4. *The General Results for Regular Polytopes.*

By applying the general formula to the six different regular polytopes we get the following table of results, where e stands for $\sqrt{5}$.

* Probably this formula giving at once the angle A for *all* the regular polytopes is new.

	p_2	p_3	p_4	a	b	c	α	β	A
C_5	3	3	3	$60°$ $\frac{1}{2}$	$70° 31' 44''$ $\frac{1}{3}$	$75° 31' 21''$ $\frac{1}{4}$	$31° 35' 11''$ $\frac{23}{27}$	$46° 34' 3''$ $\frac{11}{16}$	$14° 5' 24''$ $= 4c - \frac{8}{5}\pi$ $\cos 4c = \frac{17}{32}$
C_8	4	3	3	$90°$ 0	$90°$ 0	$90°$ 0	$90°$ 0	$90°$ 0	$90°$ 0
C_{16}	3	3	4	$60°$ $\frac{1}{2}$	$70° 31' 44''$ $\frac{1}{3}$	$120°$ $-\frac{1}{2}$	$31° 35' 11''$ $\frac{23}{27}$	$120°$ $-\frac{1}{2}$	$60°$ $\frac{1}{2}$
C_{24}	3	4	3	$60°$ $\frac{1}{2}$	$109° 28' 16''$ $-\frac{1}{3}$	$120°$ $-\frac{1}{2}$	$77° 53' 6''$ $\frac{17}{81}$	$180°$ -1	$180°$ -1
C_{120}	5	3	3	$108°$ $-\frac{1}{4}(e-1)$	$116° 33' 54''$ $-\frac{1}{e}$	$144°$ $-\frac{1}{4}(e+1)$	$169° 41' 43''$ $-\frac{11}{5e}$	$252°$ $\frac{1}{8}(3+e)$	$458° 24'$
C_{600}	3	3	5	60 $\frac{1}{2}$	$70° 31' 44''$ $\frac{1}{3}$	$164° 28' 39''$ $-\frac{1}{8}(3e+1)$	$31° 35' 11''$ $\frac{23}{27}$	$282° 23' 15''$ $\frac{1}{128}(61-15e)$	$397° 33'$ $= \frac{53}{5}\pi - 20c$ $\cos 20c = \frac{701777}{2097152}$

This table gives the angles in degrees and besides the cosine of every angle, with exception of the case A_{120} where the angle *is* mensurable and the cosine *is not.*

§ 5. *The Angle A Continued. Conclusion.*

We indicate here how the angle A of the less complicate cells, C_5, C_8, C_{16}, C_{24}, can be found without aid of Schläfli's differential formula.

The Case C_5.—By a regular truncation of C_5 at the vertices half-way up the edges, *i. e.*, such that the truncating spaces pass through the mid-points of the edges, we obtain* a polytope (10, 30, 30, 10) with ten vertices of the same kind limited by five octahedra and five tetrahedra. If we distinguish the four-dimensional angle of this new polytope from that angle A of C_5 as A', the two equations

$$2A + A' = 2(3c - \pi), \quad A + 2A' = \frac{4}{5}\pi$$

give $A = 4c - \frac{8}{5}\pi$ by elimination of A'. So we have only to prove these two equations.

The first of the two equations is found by remarking that the portion of four-dimensional space round the midpoint of an edge of C_5 inclosed by the three

* See *Proceedings*, Amsterdam, Vol. X, p. 499.

limiting spaces of C_5 passing through that edge is divided by the truncation into A' and the four-dimensional angles at that point of the two regular five-cells cut off at either side, and that its ratio to $1440°$ is equal to that of its angle β to $720°$. So we find

$$2A + A' = 2\beta = 2(3c - \pi).$$

The second of the two equations has quite another origin. If we cut the net of five-dimensional measure-polytopes by a four-dimensional space passing through a vertex normal to a diagonal, we obtain* a four-dimensional space-filling consisting of regular five-cells and of polytopes (10, 30, 30, 10) mentioned above. The 2^5 measure-polytopes round the chosen point project themselves on the diagonal in groups of

$$(1, 5, 10, 10, 5, 1);$$

of these the ten polytopes of the two groups of five are cut in five-cells, the twenty polytopes of the two groups of ten in polytopes (10, 30, 30, 10), whilst the two polytopes of the two extreme groups are left undivided. So we find that round the chosen point the four-dimensional space is filled by $10A$ and $20A'$ or

$$10A + 20A' = 1440°.$$

The values found for c_5, β_5, A_5 furnish us limits for the numbers of cells C_5 that can be arranged in four-dimensional space round a face, round an edge, round a vertex successively; these limits are evidently the integers contained in $\frac{360°}{c_5}$, $\frac{720°}{\beta_5}$, $\frac{1440°}{A_5}$, *i. e.*, 4, 15, 102. But probably the limits 15, 102 are still too large. For instance, in the analogous question about the tetrahedron in three-dimensional space $\frac{720°}{\alpha_5}$ gives the limit 22, whilst the icosahedron teaches us that we can only be sure of this, that 20 tetrahedra can concur in a common vertex, the central solid angle of the icosahedron being $36° > \alpha_5$, whilst it will probably be impossible to arrange the 20 icosahedra round a point in such a way as to leave place for a new one.

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*See *Proceedings*, Amsterdam, Vol. X, p. 688.

The Case C_8.—From the selfevident space-filling by four-dimensional measure-polytopes we deduce immediately

$$16A_8 = 1440^\circ, \text{ i. e., } A_8 = 90^\circ.$$

Here we find that it *is* possible to arrange $4C_8$ round a face, $8C_8$ round an edge, $16C_8$ round a point.

The Case C_{16}.—A C_{16} is generated by taking away from a C_8 rectangular five-cells by a regular truncation at the eight vertices of one of the two octuples of non-adjacent vertices extended so far as to remove all the original edges.* As four of the removed five-cells meet in any remaining vertex and each acute four-dimensional angle of these five-cells is an eighth part of A_{16}, the five-cell itself being a sixteenth part of C_{16}, we find

$$A_{16} = A_8 - \frac{4}{8} A_{16}, \text{ i. e., } A_{16} = 60^\circ.$$

This angle A is the smallest of the whole lot.

The same result is deduced from the well-known four-dimensional space-filling by C_{16}. By transformation of the net (C_8) into a net (C_{16}) we find indeed that it *is* possible to arrange $3C_{16}$ round a face, $6C_{16}$ round an edge, $24C_{16}$ round a point.

The Case C_{24}.—A C_{24} is obtained by joining to a C_8 at each of the eight limiting cubes a regular four-dimensional pyramid, the base of which is that cube, whilst the height is half the edge of the cube. So we get

$$A_{24} = A_8 + \frac{4}{4} A_8 = 180^\circ,$$

in accordance with the well-known four-dimensional space-filling by cells C_{24}, from which we deduce that it *is* possible to arrange $3C_{24}$ round a face, $4C_{24}$ round an edge, $8C_{24}$ round a point.

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*See my "Mehrdimensionale Geometrie", Vol II, p. 242, and *Proceedings*, Amsterdam, Vol. X, p. 536–545.

If we indicate the numbers of cells round a face, round an edge, round a point by q_2, q_1, q_0 successively, we have the results laid down in the following small table

	q_2	q_1	q_0
C_5	4	15 ?	102 ?
C_8	**4**	**8**	**16**
C_{16}	**3**	**6**	**24**
C_{24}	**3**	**4**	**8**
C_{120}	2	2	3
C_{600}	2	2	3

The figures in heavy type correspond to the cases of space-filling.